S.5 APPLIED MATHEMATICS HOLIDAY WORK 2023

Read and study the notes below, copy and write the notes in your book and after answer the questions, 2-14

TOPIC: CONTINUOUS RANDOM VARIABLES

A continuous r.v is a theoretical representation of a continuous variable such as height, mass e.t.c. The possibility space of a continuous random variable is infinite. i.e. $S = \{-\infty < x < +\infty\}$.

Thus, by definition, a continuous random variable (X) is one which takes on any value with in a given range $(a \leq x \leq b)$ of a pdf.

CONTINUOUS PROBABILITY DISTRIBUTION FUNCTIONS (PDF)

A continuous pdf is a function whose random variable, say X, takes on values within a given range. This function allocates probabilities to all of the ranges of values that random variable can take.

Properties of a continuous pdf

(1). $\int\limits_{all\ x}^{\infty} f(x)\,dx = \int_{-\infty}^{+\infty} f(x)\,dx = 1.$

(2). $f(x) \geq 0$ for all values of x.

Where $(-\infty)$ denotes the lower limit and $(+\infty)$ denotes the upper limit of the domain.

Note: we use the first property to find the unknown constants.

Finding probabilities

This involves integrating the function given for given limits of the probability.
Suppose a continuous random variable X takes on values within a interval $a \leq x \leq b$.

$$\Rightarrow P(X = a) = 0$$

This is because$X = a$ is a single value i.e. discrete. Thus,

$$P(X \leq a) = P(X = a) + P(X < a)$$
$$P(X \leq a) = 0 + P(X < a)$$
$$\Rightarrow P(X \leq a) = P(X < a)$$

So, for a continuous function: $f(x)$,

(1). $P(a \leq X \leq b) = P(a < X < b) = \int_{a}^{b} f(x)\,dx$

(2). $P(X \leq a) = P(X < a) = \int_{-\infty}^{a} f(x)\,dx$

(3). $P(X \geq a) = P(X > a) = \int_{a}^{+\infty} f(x)\,dx$

Parameters of a continuous pdf

The parameters that can be obtained from a continuous pdf include: Mean, Variance, Standard deviation, Median, Quartiles, Deciles, Percentiles and Mode.

(1). Mean or Expected value

The expected value of X, denoted as $E(X)$ *or* μ is defined as:

$$E(X) = \int_{all\ x} xf(x)\,dx = \int_{-\infty}^{+\infty} xf(x)\,dx$$

(2). Variance

$$Var(X) = E(X^2) - \{E(X)\}^2$$

where, $$E(X^2) = \int_{all\ x} x^2 f(x)\,dx = \int_{-\infty}^{+\infty} x^2 f(x)\,dx$$

Thus,

$$Var(X) = \int_{all\ x} x^2 f(x)\,dx - \left\{\int_{all\ x} xf(x)\,dx\right\}^2$$

(3). Standard deviation ($\boldsymbol{\sigma}$)

$$\sigma = \sqrt{Var(X)} = \sqrt{E(X^2) - \{E(X)\}^2}$$

Graph of a continuous pdf

This is a graph of $f(x)$ against x.

If for a given interval, $f(x)$ is a line, then: find the the starting and ending points of the line and join the points with a line.

However, if for a given interval, $f(x)$ is a curve, then: find the the starting and ending points of the curve and join the points with the appropriate curve; using the knowledge of curve sketching in Pure Mathematics.

Importance of graphs of $\boldsymbol{f(x)}$ against $\boldsymbol{x}$

(1). To find the probability between limits; say $P(a \leq X \leq b)$ by finding the area under the curve between a *and* b.

(2). To find the mode of the distribution at the highest (maximum) point of the graph.

(3). To find the constants by equating the total area under the graph to one.

(4). If the graph drawn is symmetrical, say about $x = a$, then "a" is the mean.

WORKED EXAMPLES

Qn 1: The random variable X has the probability density function

$$f(x) = \begin{cases} cx & ; \quad 0 \le x \le 1 \\ c(2-x) & ; \quad 1 \le x \le 2 \\ 0 & ; \quad otherwise \end{cases}$$

Find the contact c, the mean, median and variance. [5]

Qn 2: A continuous random variable x has a p.d.f.

$$f(x) = \begin{cases} kx & ; \quad 0 \le x < 1, \\ k(2-x) & ; \quad 1 \le x \le 2, \\ 0 & ; \quad otherwise. \end{cases}$$

(a). Sketch the graph of $f(x)$ and hence find the value of k.
(b). Find $E(X)$.
(c). Calculate $P\left(\frac{3}{4} \le x \le 1\frac{1}{2}\right)$.

Qn 3: A continuous random variable X has the probability distribution function $f(x) = kx(9 - x^2)$ for $0 \le x \le 3$, and $f(x) = 0$ otherwise , find the mode and median of the distribution. Use $k = \frac{4}{81}$.

Qn 4: A random variable X has the probability density function

$$f(x) = \begin{cases} \frac{2}{3a}(x+a) & ; \quad -a < x \le 0, \\ \frac{1}{3a}(2a-x) & ; \quad 0 \le x \le 2a, \\ 0 & ; \quad elsewhere. \end{cases}$$

Determine:
(i). the value of the constant, a,
(ii). the median of X,

Qn 5: The continuous random variable X has probability density function f given by:

$$f(x) = \begin{cases} k(x+3) & ; \quad |x| \le 3, \\ 0 & ; \quad otherwise \end{cases}$$

where k is a constant.
(i). Show that $k = \frac{1}{18}$ and sketch $f(x)$.
(ii). Find E(X).
(iii). Find q such that $P(X \le q) = \frac{1}{4}$.
(iv). Find $P(X > 1/|X| < 1.5)$.

Qn 6: The mass X kg of loaves of bread produced per hour is modeled by a continuous random variable with a probability density function given by:

$$f(x) = \begin{cases} kx^2 & ; \quad 0 \leq x \leq 2, \\ k(6-x) & ; \quad 2 \leq x \leq 6, \\ 0 & ; \quad elsewhere. \end{cases}$$

(a). Determine the value of k
(b). Sketch the graph of $f(x)$
(c). Given that a 1 kg loaf is sold at Ushs. 3500 and the running costs of baking is 2400/= per hour. Taking Y/= as the profit made in each hour, express Y in terms of x, hence find E(Y). [12]

Qn 7: A random variable X has a probability density function

$$f(x) = \begin{cases} kx(6-x)^2 & ; \quad 0 \leq x \leq 6, \\ 0 & ; \quad elsewhere. \end{cases}$$

Find the mean of X.

Qn 8: A continuous random variable X has pdf given by

$$f(x) = \begin{cases} kx & ; \quad 0 < x < 4 \\ 0 & ; \quad elsewhere \end{cases}$$

Determine:
(a). the value of k,
(b). $P(1 < X < 2)$.

Qn 9: A random Variable has a p.d.f

$$f(x) = \begin{cases} c(3x - x^2) & ; \quad 0 \leq x \leq 3 \\ 0 & ; \quad elsewhere \end{cases}$$

Find:
(i). the value of c,
(ii). $P(1 < x \leq 2)$.

Qn 10: A probability density function is given as

$$f(x) = \begin{cases} kx(4 - x^2) & ; \quad 0 \leq x \leq 2 \\ 0 & ; \quad elsewhere \end{cases}$$

Find the:
(i). value of k,
(ii). median of X,
(iii). mean of X,
(iv). standard deviation of X.

Qn 11: A continuous variable X has a pdf, $f(x)$ defined by:

$$f(x) = \begin{cases} kx^2(3 - x) & ; \quad 0 \leq x \leq 3 \\ 0 & ; \quad elsewhere \end{cases}$$

Determine:
(i). the value of the constant k.
(ii). the mean μ and the variance σ^2 of $f(x)$.
(iii). The probability that X differs from the mean by more than σ.

Qn 12: The continuous random variable X has probability density function $f(x)$ given by

$$f(x) = \begin{cases} \frac{1}{20}x^3 & 1 \leq x \leq 3 \\ 0 & otherwise. \end{cases}$$

(i). Sketch $f(x)$ for all values of x.
(ii). Calculate $E(X)$.
(iii). Show that the standard deviation of X is 0.459 to 3 decimal places.

Qn 13: The mass X kg of loaves of bread produced per hour is modeled by a continuous random variable with a probability density function given by:

$$f(x) = \begin{cases} kx^2 & ; \quad 0 \leq x \leq 2 \\ k(6-x) & ; \quad 2 \leq x \leq 6 \\ 0 & ; \quad elsewhere \end{cases}$$

(i). Determine the value of k,
(ii). Sketch the graph of $f(x)$.

Qn 14: The probability density function $f(x)$ of a random variable x is defined by

$$f(x) = \begin{cases} k(x^2+1) & -1 \leq x \leq 0 \\ k(x+1) & 0 \leq x \leq 1 \\ 0 & elsewhere \end{cases}$$

Find
(i). The value of k,
(ii). The Median of X,
(iii). Sketch the graph of $f(x)$.